

A Simple Inventory System

*Lawrence M. Leemis and Stephen K. Park, Discrete-Event Simulation: A First Course,
Prentice Hall, 2006*

Hui Chen

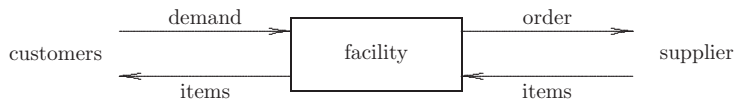
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February 8, 2017

Discrete or Continuous Variables?

- ▶ Single Server Queue
 - ▶ both arrival and service times are continuous variables
- ▶ Simple Inventory System
 - ▶ Input variables are inherently discrete

A Simple Inventory System: Conceptual Model



- ▶ Distributes items from current inventory to customers
- ▶ Order items from supplier to replenish the inventory
- ▶ Customer demand is *discrete*
 - ▶ Customers do not want a portion of an item
- ▶ Balancing *holding cost*, *shortage cost*, and *ordering cost*
- ▶ Simple
 - ▶ One type of item

Different Inventory Policies

- ▶ Transaction Reporting Policy
 - ▶ Inventory review after each transaction
 - ▶ Significant labor may be required
 - ▶ Less likely to experience shortage
- ▶ Periodic Inventory Review
 - ▶ Inventory review is periodic
 - ▶ Items are ordered, if necessary, only at review times
 - ▶ Defined by two parameters (s, S)
 - ▶ s : minimum inventory level
 - ▶ S : maximum inventory level
 - ▶ s and S are constant in time, with $0 \leq s < S$
- ▶ Assume *periodic inventory review*
- ▶ Search for (s, S) that minimize cost

Cost and Assumption

- ▶ Inventory System Costs
 - ▶ Holding cost: for items in inventory
 - ▶ Shortage cost: for unmet demand
 - ▶ Ordering cost: sum of setup and item costs
 - ▶ Setup cost: fixed cost when order is placed
 - ▶ Item cost: per-item order cost
- ▶ Additional Assumptions
 - ▶ Back ordering is possible
 - ▶ No delivery lag
 - ▶ Initial inventory level is S
 - ▶ Terminal inventory level is S

A Simple Inventory System: Specification Model

- ▶ Time begins at $t = 0$
- ▶ Review times are $t = 0, 1, 2, \dots$
 - ▶ i -th time interval begins at time $t = i - 1$ and ends at $t = i$
- ▶ I_{i-1} : inventory level at beginning of the i -th interval
- ▶ o_{i-1} : amount ordered at time $t = i - 1$ is an integer, $o_{i-1} \geq 0$
- ▶ d_i : demand quantity during the i -th interval, $d_i \geq 0$
- ▶ Inventory at end of interval can be negative
 - ▶ As a result of back-ordering

Inventory Level Consideration

- ▶ Inventory level is reviewed at $t = i - 1$
- ▶ If $l_{i-1} \geq s$, no order is placed; item If $l_{i-1} < s$, inventory is replenished to S

$$o_{i-1} = \begin{cases} 0 & l_{i-1} \geq s \\ S - l_{i-1} & l_{i-1} < s \end{cases} \quad (1)$$

- ▶ Items are delivered immediately
- ▶ At end of i -th interval, inventory diminished by d_i

$$l_i = l_{i-1} + o_{i-1} - d_i \quad (2)$$

A Simple Inventory System: Computational Model

Algorithm 1.3.1

```
 $l_0 = S;$   
 $i = 0;$   
while (more demand to process) {  
   $i ++;$   
  if ( $l_{i-1} < s$ )  
     $o_{i-1} = S - l_{i-1};$   
  else  
     $o_{i-1} = 0;$   
   $d_i = \text{GetDemand}();$   
   $l_i = l_{i-1} + o_{i-1} - d_i;$   
}  
 $n = i;$   
 $o_n = S - l_n$   
 $l_n = S;$   
return  $l_1, l_2, l_3, \dots, l_n$  and  $o_1, o_2, \dots, o_n;$ 
```


Example 1.3.1: SIS with Sample Demands

- Let $(s, S) = (20, 60)$ and apply Algorithm 1.3.1 to process $n = 12$ time intervals of operation as follows.

Note that $l_0 = S = 60$.

	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i												
inventory	l_i												

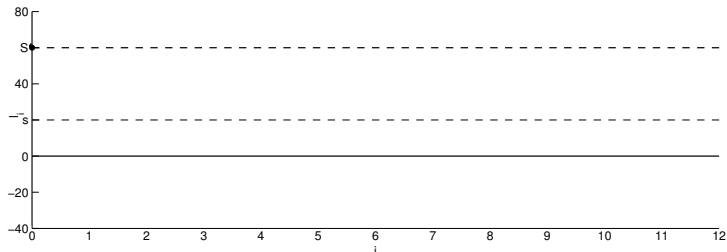


Figure: Inventory Levels

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	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0											
inventory	l_i	30											

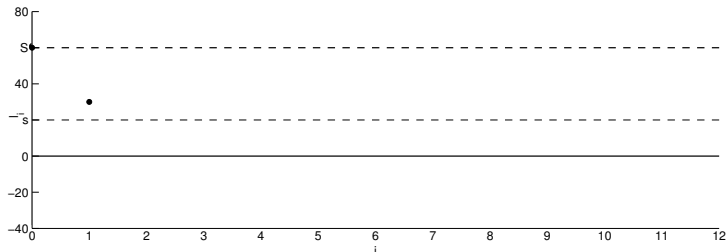


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input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45										
inventory	l_i	30	15										

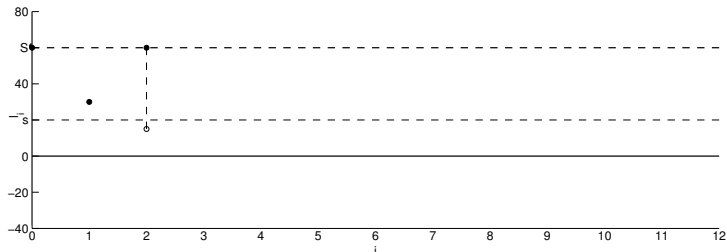


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	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45	0									
inventory	l_i	30	15	35									

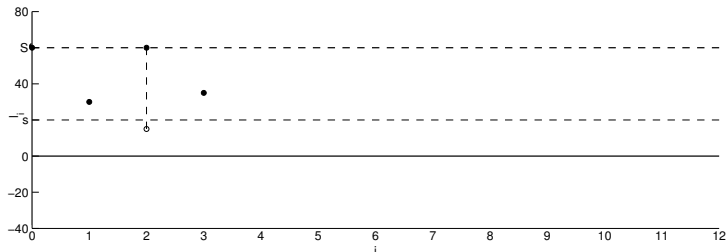


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	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45	0	0								
inventory	l_i	30	15	35	20								

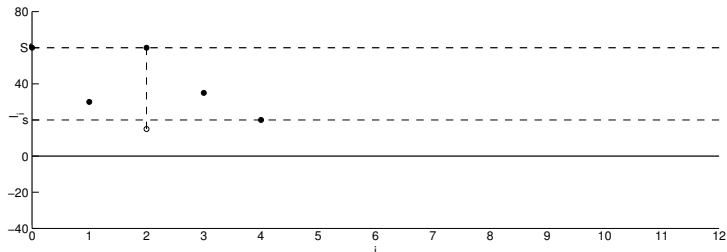


Figure: Inventory Levels

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Note that $l_0 = S = 60$.

	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45	0	0	85							
inventory	l_i	30	15	35	20	-25							

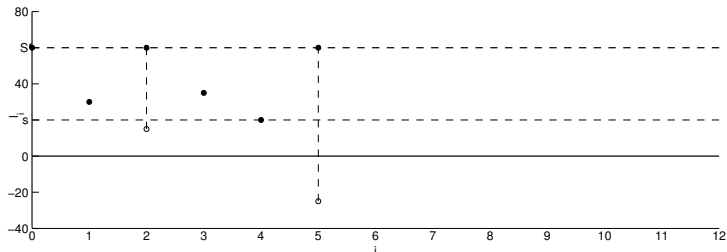


Figure: Inventory Levels

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Note that $l_0 = S = 60$.

	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i	0	45	0	0	85	...						
inventory	l_i	30	15	35	20	-25	...						

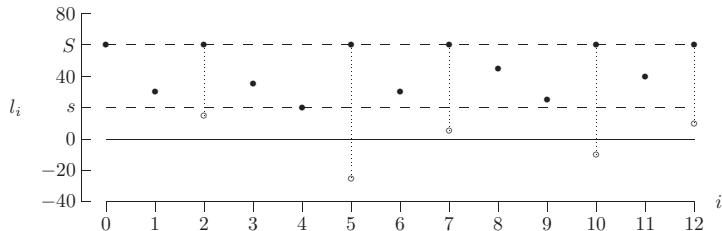


Figure: Inventory Levels

Output Statistics

- ▶ *Average demand and average order per time interval*

$$\bar{d} = \frac{1}{n} \sum_{i=1}^n d_i \quad (3)$$

$$\bar{o} = \frac{1}{n} \sum_{i=1}^n o_i \quad (4)$$

- ▶ For Example 1.3.1 Data $\bar{d} = \bar{o} = 305/12 \cong 25.42$ items per time interval

Exercise L3-1

- ▶ Let $(s, S) = (20, 60)$ and apply Algorithm 1.3.1 *by tracing the algorithm* to process $n = 12$ time intervals of operation as follows.

	i	1	2	3	4	5	6	7	8	9	10	11	12
input	d_i	30	15	25	15	45	30	25	15	20	35	20	30
order	o_i												
inventory	l_i												

- ▶ Calculate average demand. You must show the steps.
- ▶ Calculate average order. You must show the steps.

Flow Balance

- ▶ Average demand and order must be equal
- ▶ Ending inventory level is S
- ▶ Over the simulated period, all demand is satisfied
- ▶ Average “flow” of items in equals average “flow” of items out



- ▶ The inventory system is *flow balanced*

Constant Demand Rate Assumption

- ▶ Holding and shortage costs are proportional to time-averaged inventory levels
- ▶ Must know inventory level for all t
- ▶ Assume *the demand rate is constant between review times*
 - ▶ As a result, the continuous-time evolution of the inventory level is piecewise linear, as illustrated below.

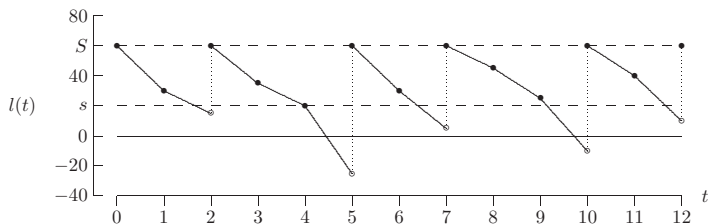


Figure: Piecewise-linear inventory levels

Inventory Level as a Function of Time

- Under condition that the demand rate (d_i) is constant between review times, then the inventory level at any time t in i -th interval is,

$$l(t) = l'_{i-1} - (t - i + 1)d_i \quad (5)$$

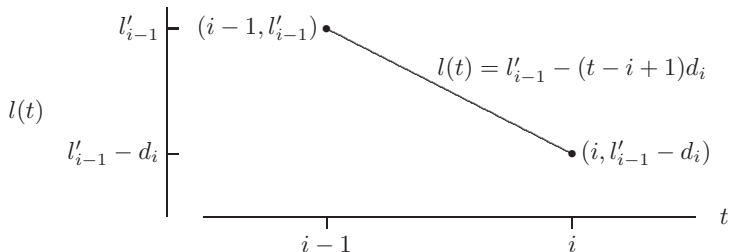


Figure: Linear inventory level in time interval i

- $l'_{i-1} = l_{i-1} + o_{i-1}$ represents inventory level after review.

Inventory Level is Not Linear!

- ▶ Inventory level at any time t is an *integer*
- ▶ $I(t)$ should be rounded to an integer value
- ▶ As a result, $I(t)$ is a stair-step, rather than linear, function
- ▶ However, it can be shown, it has no effect on the values of *Time Averaged Inventory Level* \bar{I}_i^+ and \bar{I}_i^-

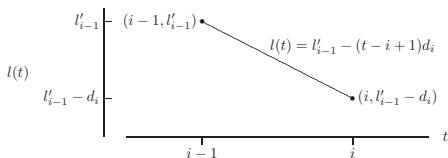


Figure: Linear inventory level in time interval i

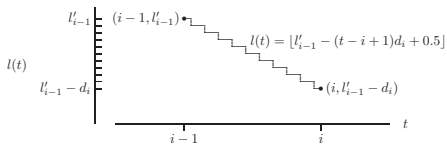


Figure: Linear inventory level in time interval i

Time Averaged Inventory Level at Time Interval i

- ▶ $I(t)$ is the basis for computing the time-averaged inventory level
 - ▶ Case 1: If $I(t)$ remains non-negative over i -th interval

$$\bar{I}_i^+ = \int_{i-1}^i I(t) dt \quad (6)$$

- ▶ Case 2: If $I(t)$ becomes negative at some time τ

$$\bar{I}_i^+ = \int_{i-1}^{\tau} I(t) dt \quad (7) \qquad \bar{I}_i^- = - \int_{\tau}^i I(t) dt \quad (8)$$

where \bar{I}_i^+ is the time-averaged *holding level* and \bar{I}_i^- is the time-averaged *shortage level*

Case 1: No Back-Ordering

- ▶ No shortage during i -th time interval if and only if $d_i \leq l'_{i-1}$

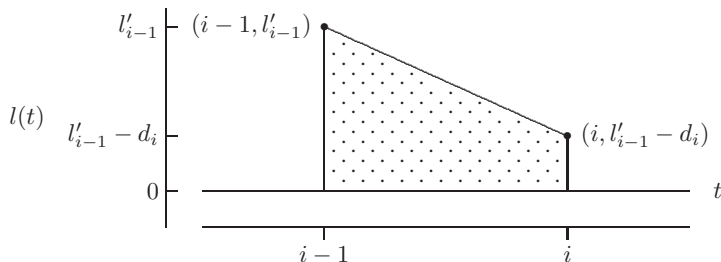


Figure: Inventory level in time interval i without back-ordering

- ▶ Time-averaged holding level: area of a trapezoid

$$\bar{l}_i^+ = \int_{i-1}^i l(t) dt = \frac{l'_{i-1} + (l'_{i-1} - d_i)}{2} = l'_{i-1} - \frac{1}{2}d_i \quad (9)$$

Case 2: Back-Ordering

- Inventory $I(t)$ becomes negative if and only if $d_i > I'_{i-1}$, i.e., at $t = \tau = i - 1 + I'_{i-1}/d_i$

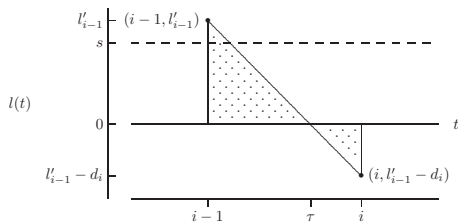


Figure: Inventory level in time interval i with back-ordering

- Time-averaged holding and shortage levels for i -th interval computed as the areas of triangles

$$\bar{I}_i^+ = \int_{i-1}^{\tau} I(t) dt = \dots = \frac{(I'_{i-1})^2}{2d_i} \quad (10) \quad \bar{I}_i^- = - \int_{\tau}^i I(t) dt = \dots = \frac{(d_i - I'_{i-1})^2}{2d_i} \quad (11)$$

\bar{l}_i^+ and \bar{l}_i^-

$$\begin{aligned} \bar{l}_i^+ &= \int_{i-1}^{\tau} l(t) dt = \frac{1}{2}[\tau - (i-1)]l'_{i-1} = \frac{(l'_{i-1})^2}{2d_i} \\ \bar{l}_i^- &= -\int_{\tau}^i l(t) dt = -\frac{1}{2}(i-\tau)(l'_{i-1} - d_i) \\ &= -\frac{1}{2}[i - (i-1 + l'_{i-1}/d_i)](l'_{i-1} - d_i) \\ &= -\frac{1}{2}(1 - l'_{i-1}/d_i)(l'_{i-1} - d_i) \\ &= -\frac{1}{2} \frac{d_i - l'_{i-1}}{d_i} (l'_{i-1} - d_i) \\ &= -\frac{(d_i - l'_{i-1})(l'_{i-1} - d_i)}{2d_i} = \frac{(d_i - l'_{i-1})^2}{2d_i} \end{aligned}$$

Time-Averaged Inventory Level

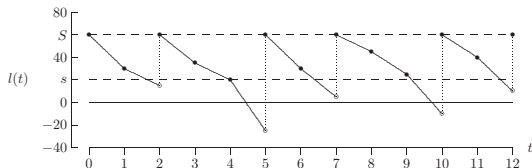


Figure: Piecewise-linear inventory levels

- ▶ *Time-averaged holding level and time-averaged shortage level*

$$\bar{l}^+ = \frac{1}{n} \sum_{i=1}^n \bar{l}_i^+ \quad (12)$$

$$\bar{l}^- = \frac{1}{n} \sum_{i=1}^n \bar{l}_i^- \quad (13)$$

- ▶ Note that time-averaged shortage level is positive
- ▶ The time-averaged inventory level is

$$\bar{l} = \frac{1}{n} \int_0^n l(t) dt = \bar{l}^+ - \bar{l}^- \quad (14)$$

Exercise L3-2

- ▶ Let $(s, S) = (20, 60)$ and process $n = 3$ time intervals of operation as follows (see slides 23, 24, and 26 for steps)

	i		1	2	3
<i>input</i>	d_i		15	45	30

- ▶ Calculate time-averaged holding level
- ▶ Calculate time-averaged shortage level

Simulation

- ▶ Create program *sis1*, a trace-driven computational model of the SIS
- ▶ Compute the output statistics
 - ▶ \bar{d} : Average demand
 - ▶ \bar{o} : Average order
 - ▶ \bar{I}^+ : Time-averaged holding level
 - ▶ \bar{I}^- : Time-averaged shortage level

and the order frequency \bar{u}

$$\bar{u} = \frac{\text{number of orders}}{n}$$

- ▶ Consistency check: compute \bar{d} and \bar{o} separately, then compare. They should be equal.

Example 1.3.4: Trace File *sis1.dat*

- ▶ Trace file *sis1.dat* contains data from $n = 100$ time intervals
- ▶ Inventory-policy parameter values (i.e., minimum & maximum inventory levels) $(s, S) = (20, 80)$
- ▶ Program *sis1* should output:

$$\bar{o} = \bar{d} = 29.29 \quad \bar{u} = 0.39 \quad \bar{T}^+ = 42.90 \quad \bar{T}^- = 0.25$$

Exercise L3-3: Implement Program `sis1`

- ▶ Develop a simulation program to implement Algorithm 1.3.1 in your favorite programming language
 - ▶ Lets call the program `ssq1`
 - ▶ In the program, compute and print *average demand*, *average order*, *time-averaged holding level*, and *time-averaged shortage level*
- ▶ Verify the program
 - ▶ Create a test case using exercises L3-1 and L3-2, and apply the test case to your program
- ▶ Use a large trace
 - ▶ Run your program using the provided “large” trace as input, observe the output

Operating Cost

- ▶ A facility's cost of operation is determined by
 - ▶ C_{item} : unit cost of new item
 - ▶ C_{setup} : fixed cost for placing an order
 - ▶ C_{hold} : cost to hold one item for one time interval
 - ▶ C_{short} : cost of being short one time for one time interval

Example 1.3.5: Case Study

- ▶ An automobile dealership that uses weekly periodic inventory review
 - ▶ The facility is the showroom and surrounding areas holding cars
 - ▶ The items are cars for sell
 - ▶ The supplier is the car manufacturer
 - ▶ The customers are the people who purchase cars from the dealership
 - ▶ Assume SIS: sells one type of car

Example 1.3.5: Results of Case Study

- ▶ Limited to a maximum of $S = 80$ cars
- ▶ Limited to a minimum of $s = 20$ cars
- ▶ Inventory reviewed every Monday
 - ▶ if inventory falls below s , order cars sufficient to restore the inventory to S
- ▶ For now, ignore delivery lag
- ▶ Then costs:
 - ▶ Item cost is $C_{item} = \$8,000$ per item
 - ▶ Setup cost is $C_{setup} = \$1,000$ per order from manufacturer
 - ▶ Holding cost is $C_{hold} = \$25$ per week
 - ▶ Shortage cost is $C_{hold} = \$700$ per week

Per-Interval Average Operating Costs

- ▶ The average operating costs per time interval are
 - ▶ $c_{item}\bar{o}$: item cost
 - ▶ $c_{setup}\bar{u}$: setup cost
 - ▶ $c_{hold}\bar{l}^+$: holding cost
 - ▶ $c_{short}\bar{l}^-$: shortage cost
- ▶ The average *total* operating cost *per time interval* is their sum

$$\bar{C}_{total} = c_{item}\bar{o} + c_{setup}\bar{u} + c_{hold}\bar{l}^+ + c_{short}\bar{l}^- \quad (15)$$

- ▶ *Total* cost of operation is the product of the average *total* operating cost *per time interval* and the number of intervals

$$C_{total} = n\bar{C}_{total} \quad (16)$$

Example 1.3.6: Operating Cost

- ▶ Use the statistics in Example 1.3.4

$$\bar{o} = \bar{d} = 29.29 \quad \bar{u} = 0.39 \quad \bar{I}^+ = 42.90 \quad \bar{I}^- = 0.25$$

and the constants in Example 1.3.5

$$c_{item} = \$8,000 \quad c_{setup} = \$1,000 \quad c_{hold} = \$25 \quad c_{short} = \$700$$

- ▶ For the dealership
 - ▶ item cost: $\$8,000 \times 29.29 = \$234,320$ per week
 - ▶ setup cost: $\$1,000 \times 0.39 = \390 per week
 - ▶ holding cost: $\$25 \times 42.40 = \$1,060$ per week
 - ▶ shortage cost: $\$700 \times 0.25 = \175 per week

Cost Minimization

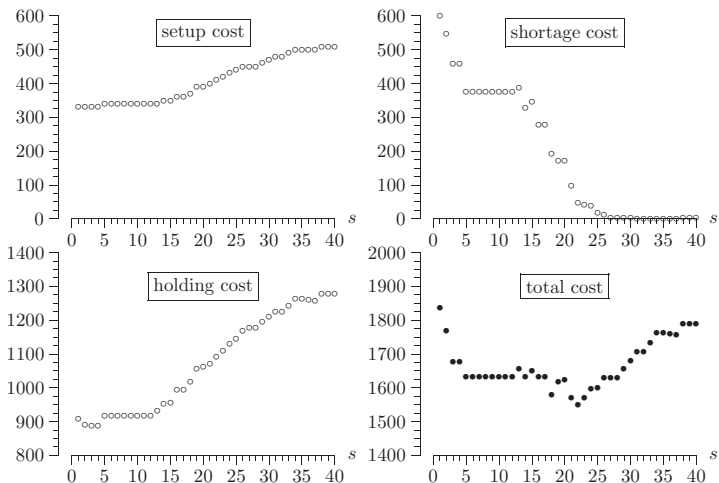
- ▶ By varying minimum inventory level s (and possibly maximum inventory level S), an *optimal policy* can be determined
- ▶ Optimal \iff minimum average total cost per time interval
- ▶ Note that $\bar{o} = \bar{d}$ and \bar{d} depends only on the demands
- ▶ Hence, item cost per time interval $c_{item}\bar{o}$ is independent of (s, S)
- ▶ Average dependent cost per time interval is the sum of these three
 - ▶ $c_{setup}\bar{u}$: average setup cost per time interval
 - ▶ $c_{hold}\bar{I}^+$: average holding cost per time interval
 - ▶ $c_{short}\bar{I}^-$: average shortage cost per time interval
 - ▶ Average dependent cost (or total cost for convenience) becomes,

$$C_{total} = c_{setup}\bar{u} + c_{hold}\bar{I}^+ + c_{short}\bar{I}^- \quad (17)$$

Experiment

- ▶ Let S be fixed, and let the demand sequence be fixed
- ▶ If s is systematically increased, we expect:
 - ▶ Average setup cost and holding cost per time interval will increase as s increases
 - ▶ Average shortage cost per time interval will decrease as s increases
 - ▶ Average dependent cost per time interval will have “U” shape, yielding an optimum (i.e., minimum cost = \$1,550 at $s = 22$)

Example 1.3.7: Optimal Periodic Inventory Review Policy



- Minimum cost = \$1,550 at $s = 22$

Exercise L3-4: sis1 and Optimizing Operating Cost

- ▶ Use the sample data (sis1.dat) and the parameters in the lecture notes, verify the optimal inventory review policy by reproducing the results in slides 35 – 37 and the graphs in slide 38.

Summary

- ▶ SIS
- ▶ Cost model and case studies