

# A Simple Inventory System

*Lawrence M. Leemis and Stephen K. Park, Discrete-Event Simulation: A First Course,  
Prentice Hall, 2006*

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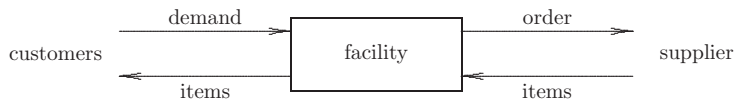
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# Discrete or Continuous Variables?

- ▶ Single Server Queue
  - ▶ both arrival and service times are continuous variables
- ▶ Simple Inventory System
  - ▶ Input variables (e.g., arrival and service times) are inherently discrete

# A Simple Inventory System



- ▶ Distributes items from current inventory to customers
- ▶ Order items from supplier to replenish the inventory
- ▶ Customer demand is discrete
  - ▶ Customers do not want a portion of an item
- ▶ Balancing *holding cost*, *shortage cost*, and *ordering cost*
- ▶ Simple
  - ▶ One type of item

# Different Inventory Policies

- ▶ Transaction Reporting Policy
  - ▶ Inventory review after each transaction
  - ▶ Significant labor may be required
  - ▶ Less likely to experience shortage
- ▶ Periodic Inventory Review
  - ▶ Inventory review is periodic
  - ▶ Items are ordered, if necessary, only at review times
  - ▶ Defined by two parameters ( $s, S$ )
    - ▶  $s$ : minimum inventory level
    - ▶  $S$ : maximum inventory level
    - ▶  $s$  and  $S$  are constant in time, with  $0 \leq s < S$
- ▶ Assume *periodic inventory review*
- ▶ Search for ( $s, S$ ) that minimize cost

# Cost and Assumption

- ▶ Inventory System Costs
  - ▶ Holding cost: for items in inventory
  - ▶ Shortage cost: for unmet demand
  - ▶ Ordering cost: sum of setup and item costs
    - ▶ Setup cost: fixed cost when order is placed
    - ▶ Item cost: per-item order cost
- ▶ Additional Assumptions
  - ▶ Back ordering is possible
  - ▶ No delivery lag
  - ▶ Initial inventory level is  $S$
  - ▶ Terminal inventory level is  $S$

# Specification Model

- ▶ Time begins at  $t = 0$
- ▶ Review times are  $t = 0, 1, 2, \dots$ 
  - ▶  $i$ -th time interval begins at time  $t = i - 1$  and ends at  $t = i$
- ▶  $I_{i-1}$ : inventory level at beginning of the  $i$ -th interval
- ▶  $o_{i-1}$ : amount ordered at time  $t = i - 1$  is an integer,  $o_{i-1} \geq 0$
- ▶  $d_i$ : demand quantity during the  $i$ -th interval,  $d_i \geq 0$
- ▶ Inventory at end of interval can be negative
  - ▶ As a result of back-ordering

# Inventory Level Consideration

- ▶ Inventory level is reviewed at  $t = i - 1$
- ▶ If  $l_{i-1} \geq s$ , no order is placed; item If  $l_{i-1} < s$ , inventory is replenished to  $S$

$$o_{i-1} = \begin{cases} 0 & l_{i-1} \geq s \\ S - l_{i-1} & l_{i-1} < s \end{cases} \quad (1)$$

- ▶ Items are delivered immediately
- ▶ At end of  $i$ -th interval, inventory diminished by  $d_i$

$$l_i = l_{i-1} + o_{i-1} - d_i \quad (2)$$

# Time Evolution of Inventory Level

## Algorithm 1.3.1

```
 $l_0 = S;$   
 $i = 0;$   
while (more demand to process) {  
     $i++;$   
    if ( $l_{i-1} < s$ )  
         $o_{i-1} = S - l_{i-1};$   
    else  
         $o_{i-1} = 0;$   
     $d_i = \text{GetDemand}();$   
     $l_i = l_{i-1} + o_{i-1} - d_i;$   
}  
 $n = i;$   
 $o_n = S - l_n$   
 $l_n = S;$   
return  $l_1, l_2, l_3, \dots, l_n$  and  $o_1, o_2, \dots, o_n;$ 
```



# Example 1.3.1: SIS with Sample Demands

- Let  $(s, S) = (20, 60)$  and apply Algorithm 1.3.1 to process  $n = 12$  time intervals of operation as follows.

Note that  $l_0 = S = 60$ .

	$i$	1	2	3	4	5	6	7	8	9	10	11	12
input	$d_i$	30	15	25	15	45	30	25	15	20	35	20	30
order	$o_i$												
inventory	$l_i$												

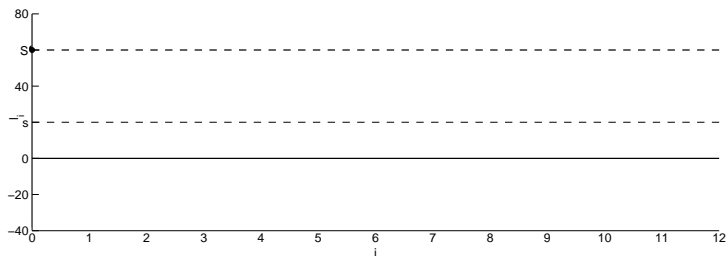


Figure : Inventory Levels

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Note that  $l_0 = S = 60$ .

	$i$	1	2	3	4	5	6	7	8	9	10	11	12
input	$d_i$	30	15	25	15	45	30	25	15	20	35	20	30
order	$o_i$	0											
inventory	$l_i$	30											

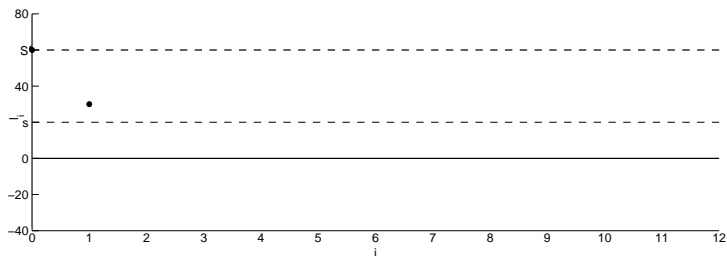


Figure : Inventory Levels

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	$i$	1	2	3	4	5	6	7	8	9	10	11	12
input	$d_i$	30	15	25	15	45	30	25	15	20	35	20	30
order	$o_i$	0	45										
inventory	$l_i$	30	15										

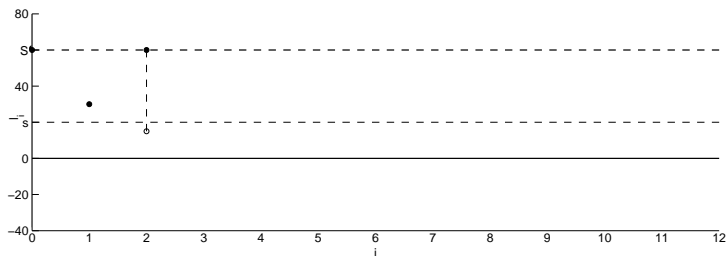


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	$i$	1	2	3	4	5	6	7	8	9	10	11	12
input	$d_i$	30	15	25	15	45	30	25	15	20	35	20	30
order	$o_i$	0	45	0									
inventory	$l_i$	30	15	35									

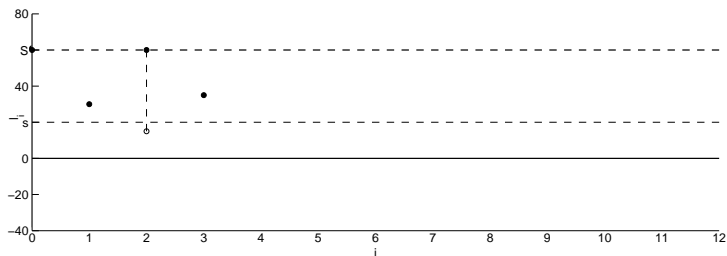


Figure : Inventory Levels

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Note that  $l_0 = S = 60$ .

	$i$	1	2	3	4	5	6	7	8	9	10	11	12
<i>input</i>	$d_i$	30	15	25	15	45	30	25	15	20	35	20	30
<i>order</i>	$o_i$	0	45	0	0								
<i>inventory</i>	$l_i$	30	15	35	20								

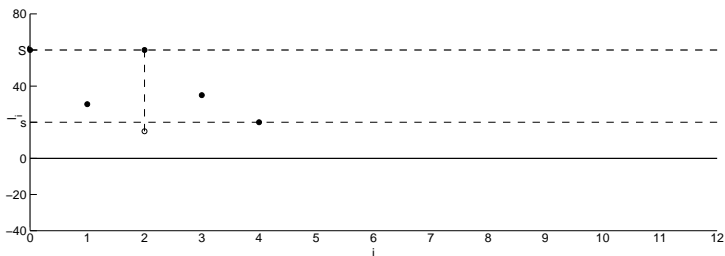


Figure : Inventory Levels

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- Let  $(s, S) = (20, 60)$  and apply Algorithm 1.3.1 to process  $n = 12$  time intervals of operation as follows.

Note that  $l_0 = S = 60$ .

	$i$	1	2	3	4	5	6	7	8	9	10	11	12
input	$d_i$	30	15	25	15	45	30	25	15	20	35	20	30
order	$o_i$	0	45	0	0	85							
inventory	$l_i$	30	15	35	20	-25							

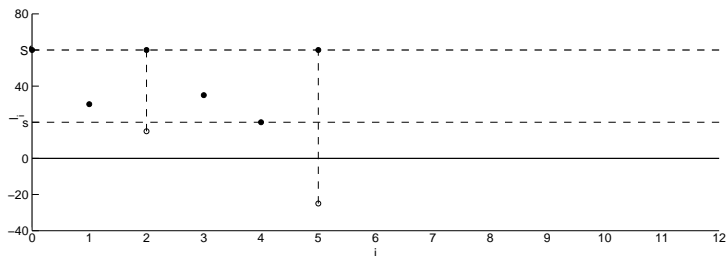


Figure : Inventory Levels

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- Let  $(s, S) = (20, 60)$  and apply Algorithm 1.3.1 to process  $n = 12$  time intervals of operation as follows.

Note that  $l_0 = S = 60$ .

	$i$	1	2	3	4	5	6	7	8	9	10	11	12
input	$d_i$	30	15	25	15	45	30	25	15	20	35	20	30
order	$o_i$	0	45	0	0	85	...						
inventory	$l_i$	30	15	35	20	-25	...						

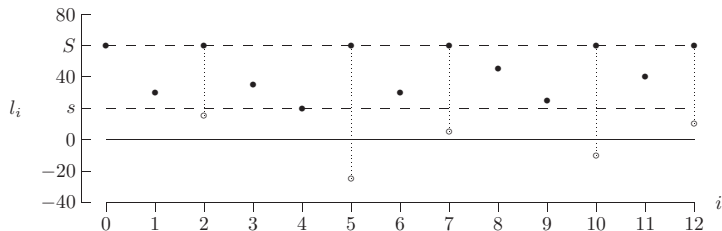


Figure : Inventory Levels

# Output Statistics

- ▶ *Average demand and average order per time interval*

$$\bar{d} = \frac{1}{n} \sum_{i=1}^n d_i \quad (3)$$

$$\bar{o} = \frac{1}{n} \sum_{i=1}^n o_i \quad (4)$$

- ▶ For Example 1.3.1 Data  $\bar{d} = \bar{o} = 305/12 \cong 25.42$  items per time interval



## Exercise L3-1

- ▶ Let  $(s, S) = (20, 60)$  and apply Algorithm 1.3.1 *by tracing the algorithm* to process  $n = 12$  time intervals of operation as follows.

	$i$	1	2	3	4	5	6	7	8	9	10	11	12
input	$d_i$	30	15	25	15	45	30	25	15	20	35	20	30
order	$o_i$												
inventory	$l_i$												

- ▶ Calculate average demand. You must show the steps.
- ▶ Calculate average order. You must show the steps.

# Flow Balance

- ▶ Average demand and order must be equal
- ▶ Ending inventory level is  $S$
- ▶ Over the simulated period, all demand is satisfied
- ▶ Average “flow” of items in equals average “flow” of items out



- ▶ The inventory system is *flow balanced*

# Constant Demand Rate Assumption

- ▶ Holding and shortage costs are proportional to time-averaged inventory levels
- ▶ Must know inventory level for all  $t$
- ▶ Assume *the demand rate is constant between review times*
  - ▶ As a result, the continuous-time evolution of the inventory level is piecewise linear, as illustrated below.

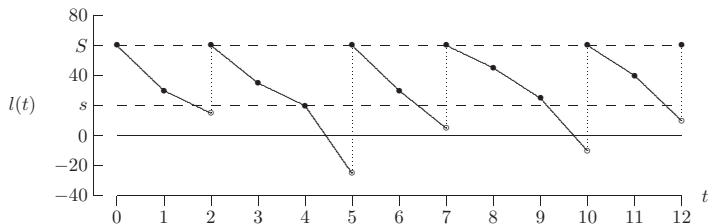


Figure : Piecewise-linear inventory levels

# Inventory Level as a Function of Time

- Under condition that the demand rate ( $d_i$ ) is constant between review times, then the inventory level at any time  $t$  in  $i$ -th interval is,

$$l(t) = l'_{i-1} - (t - i + 1)d_i \quad (5)$$

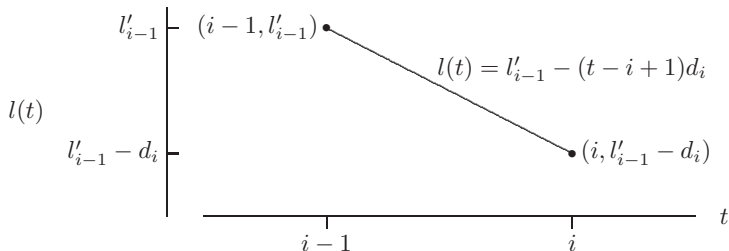


Figure : Linear inventory level in time interval  $i$

- $l'_{i-1} = l_{i-1} + o_{i-1}$  represents inventory level after review.

# Inventory Level is Not Linear!

- ▶ Inventory level at any time  $t$  is an *integer*
- ▶  $I(t)$  should be rounded to an integer value
- ▶ As a result,  $I(t)$  is a stair-step, rather than linear, function
- ▶ However, it can be shown, it has no effect on the values of *Time Averaged Inventory Level*  $\bar{I}_i^+$  and  $\bar{I}_i^-$

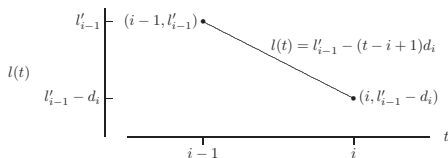


Figure : Linear inventory level in time interval  $i$

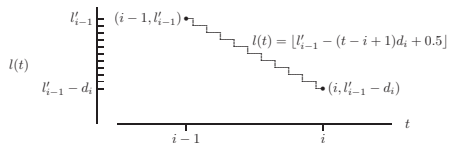


Figure : Linear inventory level in time interval  $i$

# Time Averaged Inventory Level at Time Interval $i$

- ▶  $I(t)$  is the basis for computing the time-averaged inventory level
  - ▶ Case 1: If  $I(t)$  remains non-negative over  $i$ -th interval

$$\bar{I}_i^+ = \int_{i-1}^i I(t) dt \quad (6)$$

- ▶ Case 2: If  $I(t)$  becomes negative at some time  $\tau$

$$\bar{I}_i^+ = \int_{i-1}^{\tau} I(t) dt \quad (7) \qquad \bar{I}_i^- = - \int_{\tau}^i I(t) dt \quad (8)$$

where  $\bar{I}_i^+$  is the time-averaged *holding level* and  $\bar{I}_i^-$  is the time-averaged *shortage level*

## Case 1: No Back-Ordering

- ▶ No shortage during  $i$ -th time interval if and only if  $d_i \leq l'_{i-1}$

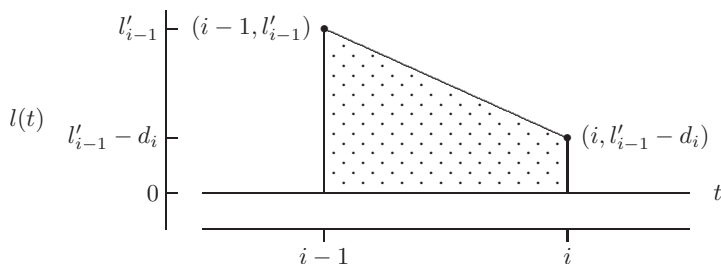


Figure : Inventory level in time interval  $i$  without back-ordering

- ▶ Time-averaged holding level: area of a trapezoid

$$\bar{l}_i^+ = \int_{i-1}^i l(t) dt = \frac{l'_{i-1} + (l'_{i-1} - d_i)}{2} = l'_{i-1} - \frac{1}{2}d_i \quad (9)$$

## Case 2: Back-Ordering

- Inventory  $I(t)$  becomes negative if and only if  $d_i > I'_{i-1}$ , i.e., at  $t = \tau = i - 1 + I'_{i-1}/d_i$

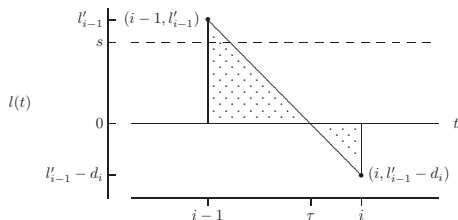


Figure : Inventory level in time interval  $i$  with back-ordering

- Time-averaged holding and shortage levels for  $i$ -th interval computed as the areas of triangles

$$\bar{I}_i^+ = \int_{i-1}^{\tau} I(t) dt = \dots = \frac{(I'_{i-1})^2}{2d_i} \quad (10) \quad \bar{I}_i^- = - \int_{\tau}^i I(t) dt = \dots = \frac{(d_i - I'_{i-1})^2}{2d_i} \quad (11)$$



$\bar{l}_i^+$  and  $\bar{l}_i^-$ 

$$\bar{l}_i^+ = \int_{i-1}^{\tau} l(t) dt = \frac{1}{2}[\tau - (i-1)]l'_{i-1} = \frac{(l'_{i-1})^2}{2d_i}$$

$$\bar{l}_i^- = -\int_{\tau}^i l(t) dt = -\frac{1}{2}(i - \tau)(l'_{i-1} - d_i)$$

$$= -\frac{1}{2}[i - (i-1 + l'_{i-1}/d_i)](l'_{i-1} - d_i)$$

$$= -\frac{1}{2}(1 - l'_{i-1}/d_i)(l'_{i-1} - d_i)$$

$$= -\frac{1}{2} \frac{d_i - l'_{i-1}}{d_i} (l'_{i-1} - d_i)$$

$$= -\frac{(d_i - l'_{i-1})(l'_{i-1} - d_i)}{2d_i} = \frac{(d_i - l'_{i-1})^2}{2d_i}$$

# Time-Averaged Inventory Level

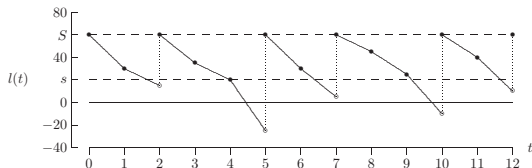


Figure : Piecewise-linear inventory levels

- ▶ *Time-averaged holding level and time-averaged shortage level*

$$\bar{l}^+ = \frac{1}{n} \sum_{i=1}^n \bar{l}_i^+ \quad (12)$$

$$\bar{l}^- = \frac{1}{n} \sum_{i=1}^n \bar{l}_i^- \quad (13)$$

- ▶ Note that time-averaged shortage level is positive
- ▶ The time-averaged inventory level is

$$\bar{l} = \frac{1}{n} \int_0^n l(t) dt = \bar{l}^+ - \bar{l}^- \quad (14)$$

## Exercise L3-2

- ▶ Let  $(s, S) = (20, 60)$  and process  $n = 3$  time intervals of operation as follows (see slides 23, 24, and 26 for steps)

	$i$		1	2	3
<i>input</i>	$d_i$		15	45	30

- ▶ Calculate time-averaged holding level
- ▶ Calculate time-averaged shortage level

# Computational Model

- ▶ Program *sis1* is a trace-driven computational model of the SIS
- ▶ Computes the statistics
  - ▶  $\bar{d}$ : Average demand
  - ▶  $\bar{o}$ : Average order
  - ▶  $\bar{l}^+$ : Time-averaged holding level
  - ▶  $\bar{l}^-$ : Time-averaged shortage level

and the order frequency  $\bar{u}$

$$\bar{u} = \frac{\text{number of orders}}{n}$$

- ▶ Consistency check: compute  $\bar{d}$  and  $\bar{o}$  separately, then compare. They should be equal.

## Example 1.3.4: Executing *sis1*

- ▶ Trace file *sis1.dat* contains data from  $n = 100$  time intervals
- ▶ Inventory-policy parameter values (i.e., minimum & maximum inventory levels)  $(s, S) = (20, 80)$
- ▶ Program outputs:

$$\bar{o} = \bar{d} = 29.29 \quad \bar{u} = 0.39 \quad \bar{I}^+ = 42.90 \quad \bar{I}^- = 0.25$$

## Exercise L3-3: sis1

- ▶ Run either C/C++ or Java program against the trace with the following inventory levels,
  - ▶  $S = 80$  and  $s = 2, 4, 6, \dots, 40$submit the result.

# Operating Cost

- ▶ A facility's cost of operation is determined by
  - ▶  $C_{item}$ : unit cost of new item
  - ▶  $C_{setup}$ : fixed cost for placing an order
  - ▶  $C_{hold}$ : cost to hold one item for one time interval
  - ▶  $C_{short}$ : cost of being short one time for one time interval

## Example 1.3.5: Case Study

- ▶ An automobile dealership that uses weekly periodic inventory review
  - ▶ The facility is the showroom and surrounding areas holding cars
  - ▶ The items are cars for sell
  - ▶ The supplier is the car manufacturer
  - ▶ The customers are the people who purchase cars from the dealership
  - ▶ Assume SIS: sells one type of car



## Example 1.3.5: Results of Case Study

- ▶ Limited to a maximum of  $S = 80$  cars
- ▶ Limited to a minimum of  $s = 20$  cars
- ▶ Inventory reviewed every Monday
  - ▶ if inventory falls below  $s$ , order cars sufficient to restore the inventory to  $S$
- ▶ For now, ignore delivery lag
- ▶ Then costs:
  - ▶ Item cost is  $c_{item} = \$8,000$  per item
  - ▶ Setup cost is  $c_{setup} = \$1,000$  per order from manufacturer
  - ▶ Holding cost is  $c_{hold} = \$25$  per week
  - ▶ Shortage cost is  $c_{hold} = \$700$  per week

# Per-Interval Average Operating Costs

- ▶ The average operating costs per time interval are
  - ▶  $c_{item}\bar{o}$ : item cost
  - ▶  $c_{setup}\bar{u}$ : setup cost
  - ▶  $c_{hold}\bar{l}^+$ : holding cost
  - ▶  $c_{short}\bar{l}^-$ : shortage cost
- ▶ The average *total* operating cost *per time interval* is their sum

$$\bar{C}_{total} = c_{item}\bar{o} + c_{setup}\bar{u} + c_{hold}\bar{l}^+ + c_{short}\bar{l}^- \quad (15)$$

- ▶ *Total* cost of operation is the product of the average *total* operating cost *per time interval* and the number of intervals

$$C_{total} = n\bar{C}_{total} \quad (16)$$

## Example 1.3.6: Operating Cost

- ▶ Use the statistics in Example 1.3.4

$$\bar{o} = \bar{d} = 29.29 \quad \bar{u} = 0.39 \quad \bar{l}^+ = 42.90 \quad \bar{l}^- = 0.25$$

and the constants in Example 1.3.5

$$c_{item} = \$8,000 \quad c_{setup} = \$1,000 \quad c_{hold} = \$25 \quad c_{short} = \$700$$

- ▶ For the dealership
  - ▶ item cost:  $\$8,000 \times 29.29 = \$234,320$  per week
  - ▶ setup cost:  $\$1,000 \times 0.39 = \$390$  per week
  - ▶ holding cost:  $\$25 \times 42.40 = \$1,060$  per week
  - ▶ shortage cost:  $\$700 \times 0.25 = \$175$  per week

# Cost Minimization

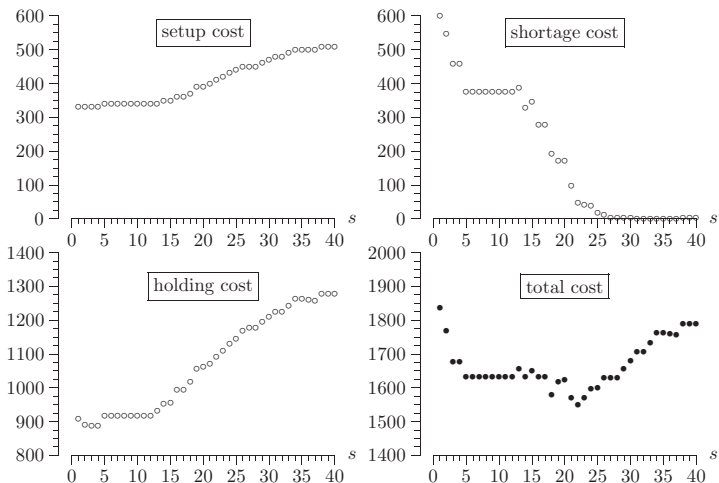
- ▶ By varying minimum inventory level  $s$  (and possibly maximum inventory level  $S$ ), an *optimal policy* can be determined
- ▶ Optimal  $\iff$  minimum average total cost per time interval
- ▶ Note that  $\bar{o} = \bar{d}$  and  $\bar{d}$  depends only on the demands
- ▶ Hence, item cost per time interval  $c_{item}\bar{o}$  is independent of  $(s, S)$
- ▶ Average dependent cost per time interval is the sum of these three
  - ▶  $c_{setup}\bar{u}$ : average setup cost per time interval
  - ▶  $c_{hold}\bar{l}^+$ : average holding cost per time interval
  - ▶  $c_{short}\bar{l}^-$ : average shortage cost per time interval
  - ▶ Average dependent cost (or total cost for convenience) becomes,

$$C_{total} = c_{setup}\bar{u} + c_{hold}\bar{l}^+ + c_{short}\bar{l}^- \quad (17)$$

# Experiment

- ▶ Let  $S$  be fixed, and let the demand sequence be fixed
- ▶ If  $s$  is systematically increased, we expect:
  - ▶ Average setup cost and holding cost per time interval will increase as  $s$  increases
  - ▶ Average shortage cost per time interval will decrease as  $s$  increases
  - ▶ Average dependent cost per time interval will have “U” shape, yielding an optimum (i.e., minimum cost = \$1,550 at  $s = 22$ )

# Example 1.3.7: Optimal Periodic Inventory Review Policy



- ▶ Minimum cost = \$1,550 at  $s = 22$

## Exercise L3-4: sis1 and Optimizing Operating Cost

- ▶ Use the sample data (sis1.dat) and the parameters in the lecture notes, verify the optimal inventory review policy by reproducing the results in slides 35 – 37 and the graphs in slide 38.
- ▶ Submit the result.

# Summary

- ▶ SIS
- ▶ Cost model and case studies