

Boolean Algebra

Hui Chen ^a

^aCUNY Brooklyn College, Brooklyn, NY, USA

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Outline

- 1 Lesson Objectives
- 2 Boolean Variables and Operations
- 3 Boolean Function
- 4 Simplifying Boolean Functions
 - Boolean Identities
 - Simplify Boolean Functions
- 5 Canonical Forms
- 6 Summary and Q&A

Acknowledgement

The content of most slides come from the authors of the textbook:

Null, Linda, & Lobur, Julia (2018). The essentials of computer organization and architecture (5th ed.). Jones & Bartlett Learning.

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Lesson Objectives

Students are expected to be able to

1. *Apply Boolean algebra and functions.*
2. Understand the relationship between Boolean logic and digital computer circuits.
3. Learn how to design simple logic circuits.
4. Understand how digital circuits work together to form complex computer systems.

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Boolean Variables

Boolean algebra is a mathematical system for the manipulation of variables that can have one of two values.

- ▶ In formal logic, these values are “true” and “false.”
- ▶ In digital systems, these values are “on” and “off,” 1 and 0, or “high” and “low.”
- ▶ The variables are called Boolean variables

Boolean expressions are created by performing operations on Boolean variables.

Manipulation: Common Boolean operators include AND, OR, and NOT.

and Truth Tables

A Boolean operator can be completely described using a truth table.

- ▶ AND. AND is also called Boolean product.
- ▶ OR. OR is called Boolean sum.
- ▶ NOT. NOT is called boolean negation. Often written as X' , \overline{X} , or $\neg X$

AND: Truth Tables

$$Z = X \text{ AND } Y = X Y$$

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

OR: Truth Tables

$$Z = X \text{ OR } Y = X + Y$$

X	Y	XY
0	0	0
0	1	1
1	0	1
1	1	1

NOT: Truth Tables

$$X' = \neg X$$

X	X'
0	1
1	0

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Boolean Function

A Boolean function has:

- ▶ at least one Boolean variable,
- ▶ at least one Boolean operator, and
- ▶ at least one input from the set $\{0, 1\}$.
- ▶ It produces an output that is also a member of the set $\{0, 1\}$.

Boolean Function: Example

The truth table for the Boolean function:

$$F(x, y, z) = xz' + y$$

To make evaluation of the Boolean function easier, the truth table contains extra columns to hold evaluations of subparts of the function

x	y	z	z'	xz'	$xz' + y$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

Rules of Precedence

As with common arithmetic, Boolean operations have rules of precedence.

The NOT operator has the highest priority, followed by AND, and then OR.

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Need to Reduce Boolean Functions

Digital computers contain circuits that implement Boolean functions.

The simpler that we can make a Boolean function, the smaller the circuit that will result.

Simpler circuits are cheaper to build, consume less power, and run faster than complex circuits.

Short List of Boolean Identities

Most Boolean identities have an AND (product) form as well as an OR (sum) form.

Identify Name	AND Form	OR Form
Identity Law	$1x = x$	$0 + x = x$
Null Law	$0x = 0$	$1 + x = 1$
Idempotent Law	$xx = x$	$x + x = x$
Inverse Law	$xx' = 0$	$x + x' = 1$
Commutative Law	$xy = yx$	$x + y = y + x$
Associative Law	$(xy)z = x(yz)$	$(x + y) + z = x + (y + z)$
Distributive Law	$x + yz = (x + y)(x + z)$	$x(y + z) = xy + xz$
Absorption Law	$x(x + y) = x$	$x + xy = x$
DeMorgan's Law	$(xy)' = x' + y'$	$(x + y)' = x'y'$
Double Complement Law	$x'' = x$	

The last group of Boolean identities are perhaps the most useful.

Boolean Function Simplification

Simplifying Boolean functions is the application of the using the Boolean identifies.

Example

Simplifying $F(x, y, z) = xy + x'z + yz$

$$\begin{aligned}F(x, y, z) &= xy + x'z + yz \\&= xy + x'z + yz1 \\&= xy + x'z + yz(x + x') \\&= xy + x'z + yzx + yzx' \\&= xy + x'z + x(yz) + (x'z)y \\&= xy + x(yz) + x'z + (x'z)y \\&= xy(1 + z) + x'z(1 + y) \\&= xy1 + x'z1 \\&= xy + x'z\end{aligned}$$

Boolean Function Complements

Sometimes it is more economical to build a circuit using the complement of a function (and complementing its result) than it is to implement the function directly.

DeMorgan's law provides an easy way of finding the complement of a Boolean function.

Example: Instead of implementing $F(x, y, z) = xy + x'z$, implement

$$\begin{aligned}F'(x, y, z) &= (xy + x'z)'\end{aligned}$$
$$\begin{aligned}&= ((xy) + (x'z))'\end{aligned}$$
$$\begin{aligned}&= (xy)'(x'z)'\end{aligned}$$
$$\begin{aligned}&= (x' + y')(x + z')\end{aligned}$$

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Equivalent Forms

Through our exercises in simplifying Boolean expressions, we see that there are numerous ways of stating the same Boolean expression.

These “synonymous” forms are logically equivalent.

Logically equivalent expressions have identical truth tables.

In order to eliminate as much confusion as possible, designers express Boolean functions in standardized or canonical form.

There are two canonical forms for Boolean expressions:

- ▶ sum-of-products and product-of-sums.

Sum-of-Products Form

Recall the Boolean product is the AND operation and the Boolean sum is the OR operation:

ANDed variables are ORed together.

Example:

$$F(x, y, z) = xy + xz + yz$$

Product-of-Sums Form

Recall the Boolean product is the AND operation and the Boolean sum is the OR operation:

ORed variables are ANDed together.

Example:

$$F(x, y, z) = (x + y)(x + z)(y + z)$$

Truth Table to Sum-of-Products

It is straightforward to convert a Boolean function to sum-of-products form using its truth table.

1. Look for rows (the values of the variables) that make the function true ($= 1$);
2. List the values of the variables that result in a true function value ($=1$), represent it as a product; and
3. Each group of variables is then ORed together.

Example

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

Example: Build Truth Table

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

x	y	z	$xz' + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Example: Look for 1's

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

x	y	z	$xz' + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Example: Construct Products

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

x	y	z	$xz' + y$	Product
0	0	0	0	
0	0	1	0	
0	1	0	1	$x'yz'$
0	1	1	1	$x'yz$
1	0	0	1	$xy'z'$
1	0	1	0	
1	1	0	1	xyz'
1	1	1	1	xyz

Example: Sum of Products!

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

x	y	z	$xz' + y$	Product
0	0	0	0	
0	0	1	0	
0	1	0	1	$x'yz'$
0	1	1	1	$x'yz$
1	0	0	1	$xy'z'$
1	0	1	0	
1	1	0	1	xyz'
1	1	1	1	xyz

Finally,

$$F(x, y, z) = x'yz' + x'yz + xy'z' + xyz' + xyz$$

But not in the simplest form!

We note that this function is not in the simplest terms!

Our aim is *only* to rewrite our function in canonical sum-of-products form.

Truth Table to Product-of-Sums

It is straightforward to convert a Boolean function to products-of-sum form using its truth table.

1. Look for rows (the values of the variables) that make the function false ($= 0$);
2. List the values of the variables that result in a false function value ($= 0$), represent it as a sum; and
3. Each group of variables is then ANDed together.

which is equivalent to

1. apply the Sum-of-Products method to F' , and
2. compute F'' .

Example: Build Truth Table

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

x	y	z	$xz' + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Example: Look for 0's

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

x	y	z	$xz' + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Example: Construct Products

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

x	y	z	$xz' + y$	Sum
0	0	0	0	$x + y + z$
0	0	1	0	$x + y + z'$
0	1	0	1	
0	1	1	1	
1	0	0	1	
1	0	1	0	$x + y' + z$
1	1	0	1	
1	1	1	1	

Example: Sum of Products!

Convert $F(x, y, z) = xz' + y$ to the Sum-of-Products form

x	y	z	$xz' + y$	Sum
0	0	0	0	$x + y + z$
0	0	1	0	$x + y + z'$
0	1	0	1	
0	1	1	1	
1	0	0	1	
1	0	1	0	$x + y'z$
1	1	0	1	
1	1	1	1	

Finally,

$$F(x, y, z) = (x + y + z)(x + y + z')(x + y' + z)$$

But not in the simplest form!

We note that this function is not in the simplest terms!

Our aim is *only* to rewrite our function in canonical Product-of-Sums form.

How about the method of $F'(x, y, z)$ '?

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Summary and Q&A

You are expected to be able to

1. *apply Boolean algebra and functions.;*

Any questions on:

- ▶ Boolean Variables and Operations
- ▶ Boolean Function
- ▶ Boolean Identities
- ▶ Simplifying Boolean Functions
- ▶ Product-of-Sums and Sum-of-Products